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ELECTRIC AND ELASTIC MULTIPOLE DEFECTS IN FINITE PIEZOELECTRIC MEDIA

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Abstract—In this paper, a piezoelectric analogy theorem is proposed, in which a piezoelectric body is represented as being composed of two fictitious bodies, an elastic body and a rigid dielectric body. An electric and elastic multipole approach for the treatment of various defects (dislocation, inhomogeneity, ...) in finite piezoelectric media is then developed. It is shown that the electric and elastic coupling effects, the boundary effects, and the defects may be considered uniformly as sources of permanent and induced electric and elastic multipoles.

1. INTRODUCTION

As early as in 1880, it was discovered by Pierre and Jacques Curie that certain crystals may, when stressed, produce an electric field, or when subjected to an electric field, deform. Such phenomena, known as piezoelectric effects, have been widely used in technology. Some recent trends are in biomechanics, for instance, the investigation of the regeneration and the remodelling properties of bone tissue by considering its elastic and electric behaviours. The piezoelectric behaviour of bone tissue is assumed to be the main causes of its bioelectric activities. A comprehensive list of works in this area may be found in the literature (Cady[1], Tiersten[2], Maugin[3], Nelson[4], Guzelsu and Demiray[5], etc.). In this paper, the physical and mechanical behaviours of various defects in finite piezoelectric media will be studied.

It is becoming known that the problems of different types of defects in various materials may be treated in a uniform way by using the concept of multipoles also sometimes called the Green's function representation (Kovács[6], Hsieh *et al.*[7], Zhou and Hsieh[8,9]). The multipole approach is based firstly on the obtainment of fundamental solutions to the basic field equations of the different materials. Unfortunately, such a fundamental Green's function solution does not yet seem to exist for piezoelectric materials not only because the basic field equations in piezoelectricity are coupled but also because piezoelectric materials are always anisotropic. To overcome these difficulties, a piezoelectric analogy theorem is first proposed, in which a piezoelectric body is represented as composed from two fictitious bodies, an elastic body and a rigid dielectric body, both with the same shape as the piezoelectric body but with different boundary conditions and different sources and loadings. By means of this theorem, an electric and elastic multipole approach for the unified treatment of the physical behaviours of various defects in finite piezoelectric media is developed.

2. BASIC FIELD EQUATIONS AND BOUNDARY CONDITIONS IN PIEZOELECTRICITY

It has been known that polarizable solid materials, when deformed, may exhibit electrical phenomena, and vice versa, piezoelectric deformation is directly proportional to the applied electric field. Such phenomena when they occur are always associated with anisotropic solids which do not have a centre of symmetry. The basic field equations in a classical linear theory of piezoelectricity[2,10] may be given as follows.

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Constitutive equations

$$t_{ij} = C_{ijkl}u_{k,l} - e_{m.ij}E_m \tag{1}$$

$$P_k = \chi_{kl}E_l + e_{k.ij}u_{i,j} \tag{2}$$

where t_{ij} is the stress tensor which is symmetric, u_k the vector of the elastic displacement, E_k the vector of the electric field, and P_k the vector of the electric polarization. The elastic moduli C_{ijkl} measured at constant (zero) electric field, the dielectric susceptibility χ_{kl} measured at constant (zero) strain, and the piezoelectric moduli $e_{m.ij}$ have the following symmetry properties, respectively

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{3a}$$

$$e_{m.ij} = e_{m.ji}, \qquad \chi_{kl} = \chi_{lk} \tag{3b}$$

in which one has used a dot to distinguish the symmetric part of the indices (ij) of the piezoelectric moduli $e_{m.ij}$ from the index (m). By introducing the electric displacement

$$D_k = P_k + \varepsilon_0 E_k \tag{4}$$

where ε_0 is the permittivity in vacuum, eqn (2) may also be written as

$$D_k = \varepsilon_{kl}E_l + e_{k.ij}u_{i,j} \tag{5}$$

in which

$$\varepsilon_{kl} = \chi_{kl} + \varepsilon_0\delta_{kl} \tag{6}$$

is the dielectric permittivity of the material.

Quasi-static Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho_e, \qquad \text{in } V \tag{7}$$

$$\nabla \times \mathbf{E} = 0, \qquad \text{or} \qquad \mathbf{E} = -\nabla\phi, \qquad \text{in } V \tag{8}$$

where ϕ is the electric potential and ρ_e is the volume density of free charges, which in general does not exist since the piezoelectric bodies, which are dielectric, are electrically neutral. However, we still keep this ρ_e term in eqn (7) for the moment.

Equilibrium equations

$$t_{ij,j} + f_i = 0, \qquad \text{in } V \tag{9}$$

where f_i is the mechanical body force.

Introducing eqns (1), (5) and (8) into eqns (7) and (9), we arrive at four field equations for the displacement vector u_k and the electric potential ϕ

$$C_{ijkl}u_{k,lj} + e_{m.ij}\phi_{,mj} + f_i = 0 \tag{10}$$

$$\varepsilon_{kl}\phi_{,lk} - e_{m.ij}u_{i,jm} + \rho_e = 0. \tag{11}$$

These differential equations should be completed by the boundary conditions. If on a part of the body ∂V_u, displacements and on the complementary part ∂V_T, tractions are prescribed

$$u_i = U_i^0(\mathbf{x}), \qquad \text{on } \partial V_u \tag{12}$$

$$t_{ij}n_j = T_i^0(\mathbf{x}), \qquad \text{on } \partial V_T, \partial V_T \cup \partial V_u = \partial V. \tag{13}$$

Suppose that on ∂V_ϕ, the electric potential and on ∂V_σ, the surface charges are given as

$$\phi = \Phi^0(\mathbf{x}), \qquad \text{on } \partial V_\phi \tag{14}$$

$$D_k n_k = \sigma(\mathbf{x}), \qquad \text{on } \partial V_\sigma, \qquad \partial V_\phi \cup \partial V_\sigma = \partial V. \tag{15}$$

The field equations, eqns (10) and (11), are coupled. The solution of the system of equations, therefore, in general poses serious mathematical difficulties. If the material contains some defects, the problem is even more difficult. In the following sections, an appropriate (multipole) method will be proposed to solve some dislocation and inhomogeneity problems in piezoelectricity. Such a method is based on an analogy theorem which we shall first derive.

3. AN ANALOGY THEOREM IN QUASI-STATIC PIEZOELECTRICITY

It is known that the Green's function method has been successfully used by many researchers in different areas to treat various problems, for instance, the multipole approach has been developed to uniformly treat the problems of different types of defects in various materials[6–9, 18]. In order to use the Green's function method for a specific material, fundamental solutions to the basic field equations must be obtained. Unfortunately, such a fundamental solution does not yet seem to exist for any practically used piezoelectric materials not only because the basic field equations in piezoelectricity are coupled but also because piezoelectric materials are always anisotropic.

Now the question is: is there any possibility of developing a multipole approach to treat some defect problems in piezoelectric materials although the corresponding Green's functions are not available? To answer this question, an analogy theorem in quasi-static piezoelectricity will first be proposed in this section. The idea is analogous to the Duhamel–Neumann analogy in quasi-static thermoelasticity.

This analogy theorem (proved in Appendix A) may be stated as follows: consider three bodies of exactly the same shape but with conditions prescribed as shown in Fig. 1. Then

$$u_i^{(1)} = u_i^{(2)}, \qquad t_{ij}^{(1)} = t_{ij}^{(2)} + e_{m.ij}^{(1)}\phi_{,m}^{(1)} \tag{16}$$

$$\phi^{(1)} = \phi^{(3)}, \qquad D_i^{(1)} = D_i^{(3)} + e_{i.kl}^{(1)}u_{k,l}^{(1)}. \tag{17}$$

This theorem means an analogy between a piezoelectric body and two fictitious bodies, an elastic body and a rigid dielectric body both with exactly the same shape as the piezoelectric body but with different boundary conditions and different sources.

Now, by means of this theorem and using the elastostatic and electrostatic reciprocal theorems, we can obtain the following system of integral equations instead of the differential field equations, eqns (10) and (11)

$$\phi(\mathbf{x}) = \int_V \rho_e G^e \,\mathrm{d}\mathbf{x}' + \int_V \varepsilon_{kl}(\phi_{,l}G^e - \phi G^e_{,l'})_{,k'}\,\mathrm{d}\mathbf{x}' - \int_V e_{k.ij}u_{i,jk}G^e\,\mathrm{d}\mathbf{x}', \qquad \text{in } V \tag{18}$$

and

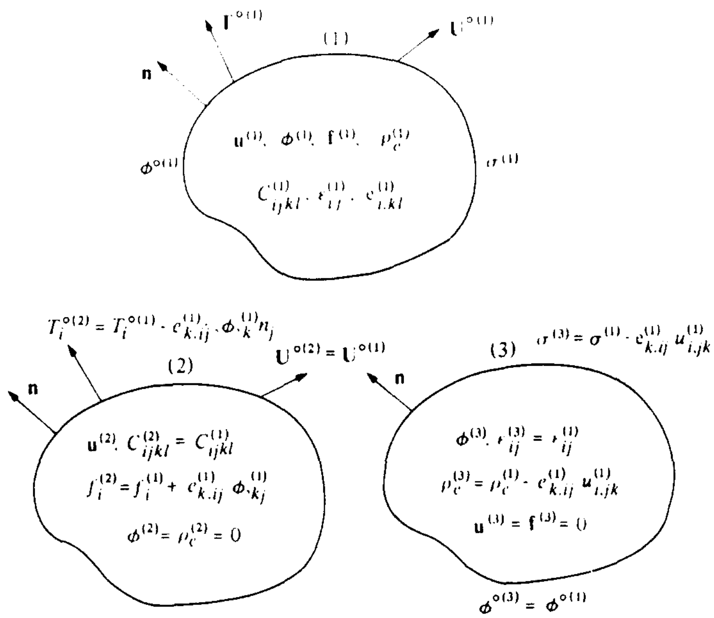

Fig. 1. An analogy between a piezoelectric body (1) and two fictitious bodies, an elastic body (2) ($\varepsilon^{(2)} = \mathbf{e}^{(2)} = 0$) and a rigid dielectric body (3) ($\mathbf{C}^{(3)} = \mathbf{e}^{(3)} = 0$).

$$u_m(\mathbf{x}) = \int_V f_i G_{im}\,\mathrm{d}\mathbf{x}' + \int_V C_{ijkl}(u_{k,l}G_{im} - u_i G_{km,l'})_{,j'}\,\mathrm{d}\mathbf{x}' + \int_V e_{l,ij}\phi_{,lj}G_{im}\,\mathrm{d}\mathbf{x}', \qquad \text{in } V \tag{19}$$

in which the Green's functions G^e and G_{im} are defined in eqns (B2) and (B9), respectively (see Appendix B). The Green's functions defined in the fictitious bodies are available for certain piezoelectric materials. For instance, we have the exact analytical solution of the Green's functions for crystals with hexagonal symmetry, such as CdS, a piezoelectric semiconductor used in delay lines and signal processing (Kröner[11] and Willis[12]). In general, various schemes to evaluate these Green's functions are also available, such as the perturbation method[13], Fredholm's technique[14] and the Fourier transform technique[15]. By means of these two equations, a multipole approach will be developed to solve some defect (dislocation, inhomogeneity, ...) problems in piezoelectric materials.

4. DISLOCATION IN PIEZOELECTRIC MEDIA AS A SOURCE OF ELECTRIC AND ELASTIC MULTIPOLES

Dislocations as sources of internal stresses often exist in crystals. The elastic fields caused by dislocations in various states of motion in bodies of various materials and geometries have been studied considerably (Hirth and Lothe[16]). The dislocation theory has been developed by many scientists to explain not only the mechanical but also the optical and electromagnetic properties of crystals (Nabarro[17]). This section is concerned with the study of the electric and elastic fields caused by a mechanical dislocation in an infinite piezoelectric medium.

According to the Volterra model, the dislocation (line defect) is defined as a part of the boundary of a slip plane S, which is embedded in the material. The strength of the dislocation is described by the Burgers vector b, which is defined as

$$b_k = u_k|_{S^+} - u_k|_{S^-}. \tag{20}$$

Now using eqn (18) in the absence of free volume charges $\rho_e = 0$, one gets

$$\phi(\mathbf{x}) = \int_S e_{k,ij} b_i G^e_{,k'} \, \mathrm{d}S'_j - \int_{V^\infty} e_{k.ij} u_i G^e_{,k'j'} \, \mathrm{d}\mathbf{x}' \tag{21}$$

and using eqn (19) in the absence of mechanical body forces, one has

$$u_m(\mathbf{x}) = -\int_S b_i C_{ijkl} G_{km,l'} \, \mathrm{d}S'_j + \int_{V^\infty} e_{n.ij} \phi G_{im,j'n'} \, \mathrm{d}\mathbf{x}' \tag{22}$$

in which one has used the continuity conditions of the electric potential and of the normal component of the electric displacement D_n across the slip surface S. The physical meaning of eqn (22) is that the elastic displacement field caused by the dislocation in the piezoelectric medium may be represented by a field which is produced by a surface distribution of permanent elastic monopoles with surface density

$$\frac{\mathrm{d}P_{kl}}{\mathrm{d}S'} = -C_{ijkl} b_i n_j, \qquad \text{on } S \tag{23}$$

and a volume distribution of the induced elastic dipoles with volume density

$$\frac{\mathrm{d}P_{ijn}}{\mathrm{d}\mathbf{x}'} = 2e_{n.ij}\phi, \qquad \text{in } V^\infty \tag{24}$$

which comes from the contribution of the electric field coupled with the elastic field. These elastic multipoles are now distributed in a corresponding elastic medium instead of the piezoelectric medium. Similarly eqn (21) means that the electric potential caused by the dislocation may be represented by a scalar field created by a surface distribution of permanent electric dipoles with surface density

$$\frac{\mathrm{d}P^e_k}{\mathrm{d}S'} = e_{k.ij} b_i n_j, \qquad \text{on } S \tag{25}$$

and a volume distribution of the induced electric quadrupoles with the volume density

$$\frac{\mathrm{d}P^e_{kj}}{\mathrm{d}\mathbf{x}'} = -2e_{k.ij} u_i, \qquad \text{in } V^\infty \tag{26}$$

which are distributed in a corresponding rigid dielectric medium instead of the piezoelectric medium. The induced elastic dipoles and the induced electric quadrupoles may be determined by a closed integral equation obtained by substituting eqn (21) into eqn (22), or inversely, the result of which may be solved by some appropriate methods (such as an iteration approach or numerical methods). As a simple example, we shall solve the dislocation problem formulated above by an iteration scheme. Suppose that the piezoelectric moduli $e_{m.ij}$ are proportional to a small parameter λ, i.e.

$$e_{m.ij} = \lambda e'_{m.ij} \tag{27}$$

and that the solutions, $\mathbf{u}(\mathbf{x})$ and $\phi(\mathbf{x})$ of the problem may be expressed as

$$u_i(\mathbf{x}) = \sum_{n=0}^{\infty} \lambda^n u_i^{(n)}(\mathbf{x}) \tag{28}$$

$$\phi(\mathbf{x}) = \sum_{n=0}^{\infty} \lambda^n \phi^{(n)}(\mathbf{x}). \tag{29}$$

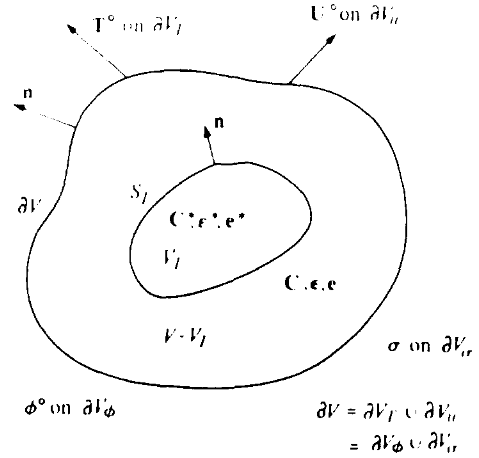

Fig. 2. A finite piezoelectric body with an inhomogeneity.

By substituting eqns (27), (28) and (29) into eqns (21) and (22), respectively, it can be shown that at the zeroth-order approximation, Volterra's classical result is recovered while at the first-order approximation, the electromechanical fields outside the singularity region caused by this mechanical dislocation may be approximately obtained as

$$u_i(\mathbf{x}) = -\int_S b_i C_{ijkl} G_{km,l'}\, \mathrm{d}S'_j \tag{30}$$

$$\phi(\mathbf{x}) = e_{k.ij}\left(\int_S b_i G^e_{,k'}\, \mathrm{d}S'_j + \int_{V^\infty - S}\left(\int_S b_m C_{mnst} G_{si,t'}\, \mathrm{d}S'_n\right) G^e_{,k'j'}\, \mathrm{d}\mathbf{x}'\right). \tag{31}$$

5. INHOMOGENEOUS INCLUSION IN FINITE PIEZOELECTRIC MEDIA AS SOURCE OF ELECTRIC AND ELASTIC MULTIPOLES

Consider a finite piezoelectric body with the elastic moduli C_{ijkl}, the piezoelectric moduli $e_{m.ij}$, and the dielectric permittivity ε_{kl}, in which there is an inhomogeneous inclusion occupying a region V_I with the elastic moduli C^*_{ijkl}, the piezoelectric moduli $e^*_{m.ij}$, and the dielectric permittivity ε^*_{kl} (see Fig. 2).

By introducing the following denotations:

$$C'_{ijkl}(\mathbf{x}) = C_{ijkl} + \Delta C_{ijkl}\alpha(\mathbf{x}) \tag{32}$$

$$\Delta C_{ijkl} = C^*_{ijkl} - C_{ijkl} \tag{33}$$

$$e'_{m.ij}(\mathbf{x}) = e_{m.ij} + \Delta e_{m.ij}\alpha(\mathbf{x}) \tag{34}$$

$$\Delta e_{m.ij} = e^*_{m.ij} - e_{m.ij} \tag{35}$$

$$\varepsilon'_{kl}(\mathbf{x}) = \varepsilon_{kl} + \Delta\varepsilon_{kl}\alpha(\mathbf{x}) \tag{36}$$

$$\Delta\varepsilon_{kl} = \varepsilon^*_{kl} - \varepsilon_{kl} \tag{37}$$

in which the indicative function $\alpha(\mathbf{x})$ is defined by

$$\alpha(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in V_1 \\ 0, & \text{otherwise} \end{cases} \tag{38}$$

then, the constitutive equations for the inhomogeneous piezoelectric medium may be written as

$$t_{ij} = C'_{ijkl}(\mathbf{x})u_{k,l} + e'_{k.ij}(\mathbf{x})\phi_{,k} \tag{39}$$

$$D_k = -\varepsilon'_{kl}(\mathbf{x})\phi_{,l} + e'_{k.ij}(\mathbf{x})u_{i,j}. \tag{40}$$

The field equations, eqns (10) and (11), may be expressed as (in the absence of external body forces and volume free charges)

$$C_{ijkl}u_{k,lj} + e_{m.ij}\phi_{,mj} + f_i^{\text{ind}} = 0 \tag{41}$$

$$\varepsilon_{kl}\phi_{,lk} - e_{k.ij}u_{i,jk} + \rho_e^{\text{ind}} = 0 \tag{42}$$

where we have introduced the induced volume charge defined by

$$\rho_e^{\text{ind}} = \Delta\varepsilon_{kl}(\phi_{,kl}\alpha(\mathbf{x}) + \phi_{,l}\alpha_{,k}(\mathbf{x})) - \Delta e_{k.ij}(u_{i,jk}\alpha(\mathbf{x}) + u_{i,j}\alpha_{,k}(\mathbf{x})) \tag{43}$$

and the induced body force defined by

$$f_i^{\text{ind}} = \Delta C_{ijkl}(u_{k,lj}\alpha(\mathbf{x}) + u_{k,l}\alpha_{,j}(\mathbf{x})) + \Delta e_{k.ij}(\phi_{,kj}\alpha(\mathbf{x}) + \phi_{,k}\alpha_{,j}(\mathbf{x})). \tag{44}$$

The physical meaning of such a manipulation may be explained as that the inhomogeneous inclusion is replaced by a distribution of induced volume charges and a distribution of induced body forces.

Now, identifying the induced volume charge ρ_e^{ind} and the induced body force f_i^{ind} with the ρ_e and f_i in eqns (18) and (19), respectively, we get

$$\phi(\mathbf{x}) = -\int_{\partial V} D_k n_k G^e \,\mathrm{d}S' - \int_{\partial V} \varepsilon_{kl}\phi G^e_{,l'} n_k \,\mathrm{d}S' + \int_V e_{k.ij}u_{i,j}G^e_{,k'}\,\mathrm{d}\mathbf{x}'$$
$$+ \int_{V_1} (-\Delta\varepsilon_{kl}\phi_{,l} + \Delta e_{k.ij}u_{i,j})G^e_{,k'}\,\mathrm{d}\mathbf{x}', \qquad \text{in } V \tag{45}$$

and

$$u_m(\mathbf{x}) = \int_{\partial V} t_{ij}n_j G_{im}\,\mathrm{d}S' - \int_{\partial V} C_{ijkl}u_i G_{km,l'} n_j \,\mathrm{d}S' - \int_V e_{k.ij}\phi_{,k}G_{im,j'}\,\mathrm{d}\mathbf{x}'$$
$$+ \int_{V_1} (\Delta C_{ijkl}u_{k,l} + \Delta e_{k,ij}\phi_{,k})G_{im,j'}\,\mathrm{d}\mathbf{x}', \qquad \text{in } V \tag{46}$$

in which the following interface conditions have been used:

(1) the electric potential ϕ and the normal component of the electric displacement D_n are continuous across the interface S_1;

(2) the elastic displacement $\mathbf{u}$ and the normal component of the stress t_n are continuous across the interface S_1.

Using the Green–Gauss theorems in eqns (45) and (46), then these may be rewritten as

$$\phi(\mathbf{x}) = -\int_{\partial V_\sigma} \sigma G^e \, \mathrm{d}S' - \int_{\partial V_\phi} \varepsilon_{kl}\Phi^0 G^e_{,l'} n_k \, \mathrm{d}S' - \int_{\partial V_\sigma} \varepsilon_{kl}\phi G^e_{,l'} n_k \, \mathrm{d}S'$$
$$- \int_{\partial V_\phi} D_k n_k G^e \, \mathrm{d}S' + \int_{\partial V_u} e_{k.ij} U_i^0 G^e_{,k'} n_j \, \mathrm{d}S' + \int_{\partial V_T} e_{k.ij} u_i G^e_{,k'} n_j \, \mathrm{d}S'$$
$$- \int_{V} e_{k.ij} u_i G^e_{,k'j'} \, \mathrm{d}\mathbf{x}' + \int_{S_1} (\Delta e_{k.ij} u_i - \Delta \varepsilon_{kj}\phi) n_j G^e_{,k'} \, \mathrm{d}S'$$
$$+ \int_{V_1} (\Delta \varepsilon_{kl}\phi - \Delta e_{k.ij} u_i) G^e_{,k'j'} \, \mathrm{d}\mathbf{x}',$$
$$\text{in } V \tag{47}$$

$$u_m(\mathbf{x}) = \int_{\partial V_T} T_i^0 G_{im} \, \mathrm{d}S' - \int_{\partial V_u} C_{ijkl} U_i^0 G_{km,l'} n_j \, \mathrm{d}S' + \int_{\partial V_u} t_{ij} n_j G_{im} \, \mathrm{d}S'$$
$$- \int_{\partial V_T} C_{ijkl} u_i G_{km,l'} n_j \, \mathrm{d}S' - \int_{\partial V_\phi} e_{k.ij} \Phi^0 G_{im,j'} n_k \, \mathrm{d}S'$$
$$- \int_{\partial V_\sigma} e_{k.ij}\phi G_{im,j'} n_k \, \mathrm{d}S' + \int_{V} e_{k.ij}\phi G_{im,j'k'} \, \mathrm{d}\mathbf{x}'$$
$$+ \int_{S_1} (\Delta C_{ijkl} u_k + \Delta e_{l.ij}\phi) n_l G_{im,j'} \, \mathrm{d}S'$$
$$- \int_{V_1} (\Delta C_{ijkl} u_k + \Delta e_{l.ij}\phi) G_{im,j'l'} \, \mathrm{d}\mathbf{x}',$$
$$\text{in } V. \tag{48}$$

The physical meaning of eqns (47) and (48) may be explained respectively as follows. Equation (47) means that the electric potential produced by an inhomogeneous inclusion in a finite piezoelectric body subjected to certain boundary conditions may be represented by a distribution of electric multipoles in a fictitious rigid uniform dielectric medium with the dielectric permittivity ε_{kl} (see Appendix B). This means that the inhomogeneous inclusion is replaced by a distribution of induced surface electric dipoles

$$\frac{\mathrm{d}P_k^e}{\mathrm{d}S'} = (\Delta\varepsilon_{kj}\phi - \Delta e_{k,ij} u_i) n_j, \qquad \text{on } S_1 \tag{49}$$

and induced volume electric quadrupoles

$$\frac{\mathrm{d}P_{kj}^e}{\mathrm{d}\mathbf{x}'} = 2(\Delta\varepsilon_{kj}\phi - \Delta e_{k,ij} u_i), \qquad \text{in } V_1. \tag{50}$$

The electric and elastic coupling effects are replaced by a distribution of permanent and induced surface electric dipoles

$$\frac{\mathrm{d}P_k^e}{\mathrm{d}S'} = e_{k.ij} U_i^0 n_j, \qquad \text{on } \partial V_u \tag{51}$$

$$\frac{\mathrm{d}P_k^e}{\mathrm{d}S'} = e_{k.ij}u_i n_j, \qquad \text{on } \partial V_T \tag{52}$$

and a distribution of induced volume electric quadrupoles

$$\frac{\mathrm{d}P_{kj}^e}{\mathrm{d}\mathbf{x}'} = -2e_{k.ij}u_i, \qquad \text{in } V. \tag{53}$$

The boundary effects are replaced by a distribution of permanent and induced surface electric charges

$$\frac{\mathrm{d}P^e}{\mathrm{d}S'} = -\sigma, \qquad \text{on } \partial V_\sigma \tag{54}$$

$$\frac{\mathrm{d}P^e}{\mathrm{d}S'} = (\varepsilon_{kl}\phi_{,l} - e_{k.ij}u_{i,j})n_k, \qquad \text{on } \partial V_\phi \tag{55}$$

and a distribution of permanent and induced surface electric dipoles

$$\frac{\mathrm{d}P_l^e}{\mathrm{d}S'} = -\varepsilon_{kl}\Phi^0 n_k, \qquad \text{on } \partial V_\phi \tag{56}$$

$$\frac{\mathrm{d}P_l^e}{\mathrm{d}S'} = -\varepsilon_{kl}\phi n_k, \qquad \text{on } \partial V_\sigma. \tag{57}$$

Similarly, eqn (48) means that the elastic displacement fields caused by the inhomogeneous inclusion in a finite piezoelectric body subjected to certain boundary conditions may be represented by a distribution of elastic multipoles in a fictitious homogeneous elastic medium with the elastic moduli C_{ijkl} (see Appendix B), that is the inhomogeneous inclusion is replaced by a distribution of induced surface elastic monopoles

$$\frac{\mathrm{d}P_{ij}}{\mathrm{d}S'} = (\Delta C_{ijkl}u_k + \Delta e_{l.ij}\phi)n_l, \qquad \text{on } S_\mathrm{I} \tag{58}$$

and a distribution of induced volume elastic dipoles

$$\frac{\mathrm{d}P_{ijl}}{\mathrm{d}\mathbf{x}'} = -2(\Delta C_{ijkl}u_k + \Delta e_{l.ij}\phi), \qquad \text{in } V_\mathrm{I}. \tag{59}$$

The electric and elastic coupling effect is replaced by a distribution of permanent and induced surface elastic monopoles

$$\frac{\mathrm{d}P_{ij}}{\mathrm{d}S'} = -e_{k.ij}\Phi^0 n_k, \qquad \text{on } \partial V_\phi \tag{60}$$

$$\frac{\mathrm{d}P_{ij}}{\mathrm{d}S'} = -e_{k.ij}\phi n_k, \qquad \text{on } \partial V_\sigma \tag{61}$$

and a distribution of induced volume elastic dipoles

$$\frac{\mathrm{d}P_{ijk}}{\mathrm{dx}'} = 2e_{k,ij}\phi, \qquad \text{in } V. \tag{62}$$

The boundary effects are replaced by a distribution of permanent and induced surface forces

$$\frac{\mathrm{d}P_i}{\mathrm{d}S'} = T_i^0, \qquad \text{on } \partial V_T \tag{63}$$

$$\frac{\mathrm{d}P_i}{\mathrm{d}S'} = (C_{ijkl}u_{k,l} + e_{m,ij}\phi_{,m})n_j, \qquad \text{on } \partial V_u \tag{64}$$

and a distribution of permanent and induced surface elastic monopoles

$$\frac{\mathrm{d}P_{kl}}{\mathrm{d}S'} = -C_{ijkl}U_i^0 n_j, \qquad \text{on } \partial V_u \tag{65}$$

$$\frac{\mathrm{d}P_{kl}}{\mathrm{d}S'} = -C_{ijkl}u_i n_j, \qquad \text{on } \partial V_T. \tag{66}$$

It has been shown that by means of an analogy theorem, the inhomogeneity problems in a finite piezoelectric body may be considered as the problems of finding a distribution of the induced electric and elastic multipoles defined respectively in a fictitious rigid uniform dielectric body and a fictitious homogeneous elastic body, in which the induced electric and elastic multipoles may be determined by a system of linear integral equations, eqns (47) and (48), that in general, may be solved by some numerical methods.

The problems of dislocations and cracks in a finite piezoelectric body may be treated in the same way[8,9].

6. CONCLUSIONS

In this paper, it is shown that the electric and elastic coupling effects, the boundary effects, and the defects may be considered uniformly as sources of permanent and induced electric and elastic multipoles. The physical quantities as interaction energy, etc....can then all be described in terms of electric and elastic multipoles. The specific multipoles are given in terms of "input". The results are given in a form particularly convenient for computational analysis.

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APPENDIX A

The verification of the piezoelectric analogy theorem proposed in Section 3 may be given as follows. Suppose that we find an elastic displacement field $\mathbf{u}^{(2)}$ in a fictitious elastic body and an electric potential field $\phi^{(3)}$ in a fictitious rigid dielectric body satisfying all the conditions described in Fig. 1 (see body (2) and body (3)), respectively, i.e.

$$t_{ij,j}^{(2)} + f_i^{(2)} = 0, \qquad t_{ij}^{(2)} = c_{ijkl}^{(2)} u_{k,l}^{(2)} \tag{A1}$$

$$\begin{aligned} u_i^{(2)} &= U_i^{0(2)}, &&\text{on } \partial V_u \\ t_{ij}^{(2)} n_j &= T_i^{0(2)}, &&\text{on } \partial V_T \end{aligned} \tag{A2}$$

and

$$D_{k,k}^{(3)} = \rho_e^{(3)}, \qquad D_k^{(3)} = -\varepsilon_{kl}^{(3)} \phi_{,l}^{(3)} \tag{A3}$$

$$\begin{aligned} \phi^{(3)} &= \Phi^{0(3)}, &&\text{on } \partial V_\phi \\ D_k^{(3)} n_k &= \sigma^{(3)}, &&\text{on } \partial V_\sigma. \end{aligned} \tag{A4}$$

We shall prove that the electromechanical fields $\mathbf{u}^{(1)}$, $\phi^{(1)}$ given by eqns (16) and (17) satisfy the piezoelectric field equations and the boundary conditions for the corresponding piezoelectric body described in Fig. 1 (see body (1)). By the constitutive equations of piezoelectricity

$$\begin{aligned} t_{ij}^{(1)} &= C_{ijkl}^{(1)} u_{k,l}^{(1)} + e_{m,ij}^{(1)} \phi_{,m}^{(1)} \\ D_k^{(1)} &= -\varepsilon_{kl}^{(1)} \phi_{,l}^{(1)} + e_{k,ij}^{(1)} u_{i,j}^{(1)} \end{aligned}$$

and using eqns (16) and (17) and eqns (A1) and (A3), we get

$$\begin{aligned} t_{ij,j}^{(1)} &= -f_i^{(2)} + e_{m,ij}^{(1)} \phi_{,mj}^{(1)} = -f_i^{(1)}, &&\text{in } V \\ D_{k,k}^{(1)} &= \rho_e^{(3)} + e_{k,ij}^{(1)} u_{i,jk}^{(1)} = \rho_e^{(1)}, &&\text{in } V. \end{aligned}$$

Using eqns (A2) and (A4), we have

$$\begin{aligned} u_i^{(1)} &= U_i^{0(2)} = U_i^{0(1)}, &&\text{on } \partial V_u \\ t_{ij}^{(1)} n_j &= T_i^{0(2)} + e_{k,ij}^{(1)} \phi_{,k}^{(1)} n_j = T_i^{0(1)}, &&\text{on } \partial V_T \end{aligned}$$

and

$$\begin{aligned} \phi_i^{(1)} &= \Phi^{0(2)} = \Phi^{0(1)}, &&\text{on } \partial V_\phi \\ D_k^{(1)} n_k &= \sigma^{(3)} + e_{k,ij}^{(1)} u_{i,j}^{(1)} n_j = \sigma^{(1)}, &&\text{on } \partial V_\sigma. \end{aligned}$$

It is shown that the electromechanical fields given by eqns (16) and (17) are the solutions of the piezoelectric body subjected to the loadings and boundary conditions described in Fig. 1. Q.E.D.

APPENDIX B. STATIONARY ELECTRIC AND ELASTIC MULTIPOLES

Consider M point charges located in a small volume centred at $\mathbf{x}'$ of an infinite uniform dielectric medium with dielectric permittivity ε_{ij}. The resulting electric potential may be obtained as

$$\phi(\mathbf{x}) = \sum_{\alpha=1}^{M} q^\alpha G^e(\mathbf{x}, \mathbf{x}' + \mathbf{d}^\alpha) \tag{B1}$$

where $\mathbf{x}' + \mathbf{d}^\alpha$ is the position vector of the αth point charge q^α, and G^e is the Green's function satisfying

$$\varepsilon_{kl} G^e_{,lk}(\mathbf{x}, \mathbf{x}') + \delta^e(\mathbf{x} - \mathbf{x}') = 0 \tag{B2}$$

which may be regarded physically as the electric potential produced by a unit point charge located at $\mathbf{x}'$.

Expanding the Green's function in a Taylor series about $(\mathbf{x}, \mathbf{x}')$, we get

$$\phi(\mathbf{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} P^e_{s_1 \ldots s_k} G^e_{,s_1 \ldots s_k}(\mathbf{x}, \mathbf{x}') \tag{B3}$$

where we have introduced the electric multipoles of order k

$$P^e_{s_1 \ldots s_k} = \sum_{\alpha=1}^{M} q^\alpha d^\alpha_{s_1} \ldots d^\alpha_{s_k}. \tag{B4}$$

For $k = 0$, we get the resultant charge of the point charge array

$$P^e = \sum_{\alpha=1}^{M} q^\alpha. \tag{B5}$$

For $k = 1$, we get the electric dipole

$$P^e_i = \sum_{\alpha=1}^{M} q^\alpha d^\alpha_i \tag{B6}$$

..., etc.

Similarly, consider N point body forces acting in a small volume centred at $\mathbf{x}'$ of an infinite homogeneous elastic medium with elastic moduli C_{ijkl}. The resulting displacement fields may be obtained as (Siems[18])

$$\begin{aligned} u_m(\mathbf{x}) &= \sum_{\alpha=1}^{N} f^\alpha_j G_{mj}(\mathbf{x}, \mathbf{x}' + \mathbf{d}^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} P_{js_1 \ldots s_n} G_{mj,s_1 \ldots s_n}(\mathbf{x}, \mathbf{x}') \end{aligned} \tag{B7}$$

where the elastic multipoles of order n are defined as

$$P_{js_1 \ldots s_n} = \sum_{\alpha=1}^{N} f^\alpha_j d^\alpha_{s_1} \ldots d^\alpha_{s_n}. \tag{B8}$$

If the resultant force of the point force array is zero, we have $P_j = 0$, and call P_{js}, P_{jst}, ... elastic monopole, elastic dipole, ..., respectively. The Green's function G_{im} satisfying the equations

$$C_{ijkl} G_{im,lj}(\mathbf{x}, \mathbf{x}') + \delta(\mathbf{x} - \mathbf{x}')\delta_{km} = 0 \tag{B9}$$

may be regarded physically as the displacement along the x_i-axis at $\mathbf{x}$ produced by a unit point body force applied along the x_m-axis at $\mathbf{x}'$.